Risk measures and multivariate extensions of Breiman's Theorem

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Abstract

Modeling insurance risks is a task that received an increasing attention because of Solvency Capital Requirements. The ruin probability has become a standard risk measure to assess regulatory capital. In this paper we focus on discrete time models for finite time horizon. Several results are available in the literature allowing to calibrate the ruin probability by means of the sum of the tail probabilities of individual claim amounts. The aim of this work is to obtain asymptotics for such probabilities under multivariate regularly variation and, more precisely, to derive them from Breiman's Theorem extensions. We thus exhibit new situations where the ruin probability admits computable equivalents. Consequences are also derived in terms of the Value-at-Risk.

Keywords:
Discrete time model; Ruin probability; Value-at-Risk; Multivariate regular variation; Dependent risks.

1 Introduction

Let x be a nonnegative real number, $(\rho_i)_{i\in\mathbb{N}^*}$ and $(X_i)_{i\in\mathbb{N}^*}$ be two sequences of random variables (r.v.), and define the sequence $(R_i)_{i\in\mathbb{N}}$ by the recursive equation

$$R_0 = x,\ R_i = R_{i-1}(1+\rho_i) - X_i, \quad i \in \mathbb{N}^* \ . \tag{1}$$

This model has been used to model insurance risks. In such a context, x stands for the initial capital, $\rho_i \in (-1,\infty)$ for the random interest rate for the ith period and X_i for the total claim amount minus the total premium incomes, so that R_i represents the discounted surplus computed from period 0 to i. In this paper we focus on discrete time models for fixed finite time horizon d.

The ruin probability has become a standard risk measure to assess regulatory capital in insurance. For the model (1), the ruin probability within finite time horizon $[0,d]$ and initial capital reserve x is defined by

$$\psi(x,d) = \mathbb{P}\left(\min_{1\le k\le d} R_k < 0 \ \mid\ R_0 = x\right) \ .$$

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Introducing the so-called discount factor from period i to 0, denoted by $Y_i = \prod_{j=1}^{i}(1+\rho_j)^{-1}$, one can re-express the discrete time risk model as

$$R_0 = x,\ R_i = Y_i^{-1}\left(x - \sum_{j=1}^{i} X_j Y_j\right), \quad i \in \mathbb{N}^* . \tag{2}$$

Hence, the ruin probability can be written as

$$\psi(x,d) = \mathbb{P}\left(\max_{1\le k\le d} \sum_{i=1}^{k} X_i Y_i > x\right) . \tag{3}$$

Several results exist on the limiting behavior of $\psi(x,d)$, especially in the case of an initial reserve x tending to infinity and when the distribution function of the X_i's is subexponential and sometimes heavy tailed. Most of them are stated under independence between the components of $\mathbf{X} = (X_1,\ldots,X_d)$: When $\mathbf{Y} = (Y_1,\ldots,Y_d)$ is deterministic, we know these asymptotics from Embrechts and Veraverbeke [1982], Sgibnev [1996], Embrechts et al. [1997], Ng et al. [2002] and Zhu and Gao [2008] among others. The case where the Y_i's are bounded r.v. has been studied by Tang and Tsitsiashvili [2003a,b]. Models governed by a specific dependence structure of the random vector $\mathbf{Y}$ have been proposed by Nyrhinen [1999], Cai [2002] and Chen and Su [2006]. Recent extensions to any dependence structure of $\mathbf{Y}$ can be found in Goovaerts et al. [2005], Wang et al. [2005] and Wang and Tang [2006] where the results are proven under a moment condition.

In some situations, the assumption of independence between the claim amounts X_i might be unrealistic. Cossette and Marceau [2000] considered special models of dependency. Allowing general dependence among the X_i's but keeping them independent from the nonnegative weights Y_i's, Zhang et al. [2009] derived approximations for $\psi(x,d)$ under the assumptions that the X_i's have (extended) regularly varying tail and are asymptotically independent. Roughly speaking, the latter notion means that the joint upper tail of two claim amounts is negligible compared with each univariate tail. More precisely, Zhang et al. [2009] established the following equivalences, as x tends to infinity:

$$\psi(x,d) \sim \mathbb{P}\left(\sum_{i=1}^{d} X_i Y_i > x\right) \sim \sum_{i=1}^{d} \mathbb{P}\left(X_i Y_i > x\right) . \tag{4}$$

In their context, the first equivalence in (4) is a consequence of the assumption that the left tails of the X_i's are lighter than their right tails (see (9) below for a precise definition) and of the positivity of the weights Y_i. The second equivalence in (4), which follows from the asymptotic independence of the X_i's, has also its own interest. Indeed, several papers are concerned with how the tail of the marginal distribution of an individual summand influences the asymptotic behavior of the sum. Barbe et al. [2006] extended the equivalence between the tail of a sum and the sum of the tails, already obtained by Wüthrich [2003], Alink et al. [2004, 2005], to a broader class of dependence structures using multivariate extreme value theory. Recently, Kortschak and Albrecher [2009] treated the case of non-identically distributed and not necessarily positive random variables.

The aim of this paper is to establish, in various contexts of dependence, asymptotics of the ruin probability and the Value-at-Risk. Our generalizations of (4) are in two directions: we allow dependence between the claim amounts X_i and the weights Y_i; and we relax the assumption of asymptotic independence of the claim amounts. These results are obtained under the key assumption that $\mathbf{X}$ is Multivariate Regularly Varying (MRV) at infinity, see for instance Resnick [2007, chapter 6]. More precisely, the paper highlights the fact that our extensions are valid as soon as the random vector $(X_1Y_1, \ldots, X_dY_d)$ is also MRV, which is known as a Breiman's type result.

The rest of the paper is organized as follows: Section 2 contains notation and preliminary results. Several generalizations of Breiman's Theorem are stated in Section 3. Section 4 takes benefit of these extensions to derive asymptotics of risk measures as ruin probability and Value-at-Risk. A brief conclusion is given in Section 5. The proofs are postponed until Section 6.

2 Notation, definitions and preliminary results

We start this section with some notation. Let $\mathbf{x} \cdot \mathbf{y}$ define the componentwise product of two vectors $\mathbf{x} = (x_1, \ldots, x_d)^T$ and $\mathbf{y} = (y_1, \ldots, y_d)^T \in \mathbb{R}^d$, i.e.

$$\mathbf{x} \cdot \mathbf{y} = (x_1y_1, \ldots, x_dy_d)^T$$

and let

$$\mathbf{z}^{-1} = (z_1^{-1}, \ldots, z_d^{-1})^T$$

be the componentwise inverse of $\mathbf{z} \in [0, \infty]^d$. We define the inverse image of a set A for $\mathbf{y} \in [0, \infty]^d$ by

$$\mathbf{y}^{-1} \cdot A = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{y} \cdot \mathbf{x} \in A\} = \{(y_1^{-1}x_1, \ldots, y_d^{-1}x_d)^T, \mathbf{x} \in A\} .$$

Let $\mathbf{X}$ be a d-dimensional random vector. We say that $\mathbf{X}$ is *multivariate regularly varying* if there exists a non-null Radon measure ν on $\mathcal{B}_d$, the Borel σ-field of $\mathbb{E}_d = [-\infty, \infty]^d \setminus \{\mathbf{0}\}$, such that $\nu((-\infty, \infty)^d \setminus \{\mathbf{0}\}) > 0$ and a normalizing function a (with $a(t) \to \infty$) such that

$$t\mathbb{P}(\mathbf{X}/a(t) \in \cdot) \xrightarrow{v} \nu , \tag{5}$$

as t tends to infinity, where $\xrightarrow{v}$ refers to vague convergence on $\mathcal{B}_d$. Recall that a Radon measure on $\mathcal{B}_d$ is a measure that is finite on each compact set of $\mathbb{E}_d$, and that a set $A \subset \mathbb{E}_d$ is relatively compact if it is bounded away from zero. Recall also that a sequence of measures ν_n converges vaguely to ν on $\mathcal{B}_d$ if $\int_{\mathbb{E}_d} f \, \mathrm{d}\nu_n$ converges to $\int_{\mathbb{E}_d} f \, \mathrm{d}\nu$ for any function f compactly supported on $\mathbb{E}_d$, or equivalently if $\nu_n(K)$ converges to $\nu(K)$ for each relatively compact set K of $\mathbb{E}_d$ such that $\nu(\partial K) = 0$.

The limit measure ν is necessarily homogeneous, i.e. $\nu(tK) = t^{-\alpha}\nu(K)$ for some $\alpha > 0$ and all relatively compact Borel set K of $\mathbb{E}_d$. The function a is regularly varying with index $1/\alpha$. We shall write $\mathbf{X} \in \mathrm{MRV}(\alpha, a, \nu)$ if (5) holds or simply MRV if no confusion can arise.

In the convergence (5) one may choose the normalizing function a such that

$$\lim_{t\to\infty} t\,\mathbb{P}(\|\mathbf{X}\| > a(t)) = 1 , \tag{6}$$

where $\|\cdot\|$ denotes some norm on $\mathbb{R}^d$. Given the choice of a norm on $\mathbb{R}^d$, a polar representation of the measure ν can be obtained. This result is due to de Haan and Resnick [1977], refer for instance to Resnick [2007, Theorem 6.1 p. 173 and Section 6.5.5 p. 201]. Let $\mathbb{S}^{d-1}_{\|\cdot\|}$ denote the unit sphere of $\mathbb{R}^d$ relatively to the norm $\|\cdot\|$. Then $\mathbf{X}$ is MRV if and only if there exists a measure $H_{\|\cdot\|}(\cdot)$ on $\mathbb{S}^{d-1}_{\|\cdot\|}$, a positive real α, and a normalizing function a (with $a(t) \to \infty$) such that

$$t\mathbb{P}\left(\left(\frac{\|\mathbf{X}\|}{a(t)}, \frac{\mathbf{X}}{\|\mathbf{X}\|}\right) \in \cdot\right) \xrightarrow{v} (\eta_\alpha \times H_{\|\cdot\|})(\cdot) \;, \tag{7}$$

as t tends to infinity, where vague convergence holds on $(0,\infty] \times \mathbb{S}^{d-1}_{\|\cdot\|}$ and η_α denotes the Radon measure on $(0,\infty]$ defined by $\eta_\alpha(x,\infty] = x^{-\alpha}$. The measure $H_{\|\cdot\|}$ is called the *spectral measure*, or *angular measure*, and the choice of the normalizing function in (6) implies that $H_{\|\cdot\|}$ is a probability measure on $\mathbb{S}^{d-1}_{\|\cdot\|}$. The link between the limit measure ν and the spectral measure $H_{\|\cdot\|}$ can be explicited via the following decomposition, for any $A \in \mathcal{B}_d$,

$$\nu(A) = \int_{T(A)} \frac{\alpha \mathrm{d}r}{r^{\alpha+1}} H_{\|\cdot\|}(\mathrm{d}\mathbf{w}) \;, \tag{8}$$

where $T(\mathbf{z}) = (\|\mathbf{z}\|, \mathbf{z}/\|\mathbf{z}\|)$ for any $\mathbf{z} \in \mathbb{E}_d$ and $T(A)$ is the usual image of A by T.

We will use the following property in the sequel, and sketch the proof in Section 5.

Lemma 1. *If $\mathbf{X}$ is MRV(α, a, ν) on $\mathbb{R}^d$ and if the left tail of each component X_i is lighter than its right tail, i.e.*

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_i < -x)}{\mathbb{P}(X_i > x)} = 0 \;, \quad 1 \le i \le d \;, \tag{9}$$

then the measure ν is concentrated on $[0,\infty]^d \setminus \{\mathbf{0}\}$, the spectral measure $H_{\|\cdot\|}$ of $\mathbf{X}$ is concentrated on $\mathbb{S}^{d-1}_{\|\cdot\|} \cap [0,\infty]^d$ and

$$\lim_{x\to\infty} \frac{\mathbb{P}(\max_{1\le j\le d} \sum_{i=1}^{j} X_i > x)}{\mathbb{P}(\sum_{i=1}^{d} X_i > x)} = \lim_{x\to\infty} \frac{\mathbb{P}(\sum_{i=1}^{d} X_i^+ > x)}{\mathbb{P}(\sum_{i=1}^{d} X_i > x)} = 1 \;.$$

Remark 1. Following Kortschak and Albrecher [2009, Lemma 3.1] the condition (9) may be replaced by

$$\mathbb{P}(X_i > a, X_j > b) \ge \mathbb{P}(X_i > a)\mathbb{P}(X_j > b) \text{ for all}(a,b) \in \mathbb{R}^2 \text{ and } 1 \le i < j < n. \tag{10}$$

Throughout this paper, we will choose the ℓ_1-norm $\|\cdot\|_1$ defined by

$$\|\mathbf{x}\|_1 = \sum_{j=1}^{d} |x_i| \;,$$

and we will denote $\mathbb{S}^{d-1}_1$ and H_1 the corresponding unit sphere and spectral measure. The motivation of these choices for the normalizing function and the norm on $\mathbb{R}^d$ is that if $\mathbf{X}$ is MRV and satisfies the assumptions of Lemma 1, then on one hand, one has for all $x > 0$,

$$\lim_{t\to\infty} t\mathbb{P}\left(\sum_{i=1}^{d} X_i > a(t)x\right) = x^{-\alpha} \;. \tag{11}$$

For any d-dimensional vector $\mathbf{X}$, let $Q(\mathbf{X})$ denote the limit, when it exists,

$$Q(\mathbf{X}) = \lim_{x\to\infty} \frac{\mathbb{P}(\sum_{i=1}^d X_i > x)}{\sum_{j=1}^d \mathbb{P}(X_j > x)} .$$

Similar limits to $Q(\mathbf{X})$ have been considered in the literature. The following result can be found in several references and under different assumptions. We refer for instance to Alink et al. [2004], Barbe et al. [2006], Kortschak and Albrecher [2009], Embrechts et al. [2009b,c]. The unique assumption needed is multivariate regular variation, as proven in Section 6. The components of a regularly varying random vector are called *asymptotically independent* if its spectral measure (with respect to any norm) is concentrated on the axes.

Lemma 2. *If $\mathbf{X}$ is a d-dimensional multivariate regularly varying random vector with index $\alpha > 0$ such that (9) holds, then $Q(\mathbf{X})$ exists and*

$$d^{-(1-\alpha)_+} \leq Q(\mathbf{X}) \leq d^{(\alpha-1)_+} . \tag{12}$$

If the components of $\mathbf{X}$ are asymptotically independent, then $Q(\mathbf{X}) = 1$.

Remark 2. Note that these bounds are universal and (obviously) does not depend on any particular choice of norm on $\mathbb{R}^d$, even though the proof we give below makes use of the ℓ_1-norm. For analogous comments on the choice of norm, we refer to the remark in Mainik and Rüschendorf [2009] after the representation (13) therein or to [Embrechts et al., 2009a, Proposition 4.6].

Remark 3. Note that contrary to Barbe et al. [2006] and [Embrechts et al., 2009a, Corollary 4.2], we do not assume that the random variables X_i are positive nor that they have the same marginal distribution. This implies in particular that the usual standardization $\int w_i H_1(\mathrm{d}w) = 1/d$ (Barbe et al. [2006, Equation (11)] or Beirlant et al. [2004, p. 260]) does not hold here, but is not needed. Kortschak and Albrecher [2009] also considered such generalizations to non positive or non identically distributed marginals. However, the bounds given in Lemma (12) are new in this context.

Remark 4. If $\mathbf{X}$ is multivariate regularly varying with index $\alpha > 0$, for any fixed norm on $\mathbb{R}^d$, $Q(\mathbf{X})$ depends only on α and on the associated spectral measure H. Thus, with an abuse of notation, we can denote $Q(\mathbf{X}) = Q(\alpha, H)$. The properties of $Q(\alpha, H)$ have first been investigated by [Barbe et al., 2006, Proposition 2.2]. For a given α, the upper and lower bounds in (12) are achieved by independent or fully dependent components, and for a given spectral measure H, $Q(\alpha, H)$ is increasing in α. See also Kortschak and Albrecher [2009] and Embrechts et al. [2009b,c].

3 Multivariate extensions of Breiman's Theorem

Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors in $\mathbb{R}^d$. A Breiman's type result consists in obtaining sufficient conditions for the vector $\mathbf{Y} \cdot \mathbf{X}$ to be MRV. We start by recalling a particular case of Basrak et al. [2002, Proposition A.1] who prove the multivariate regular variation of $\mathbf{MX}$ where $\mathbf{M}$ is a random matrix independent of $\mathbf{X}$. Denote by $\nu_{\mathbf{Y}}$ the measure defined on $\mathbb{E}_d$ by

$$\nu_{\mathbf{Y}}(K) = \mathbb{E}[\nu(\mathbf{Y}^{-1} \cdot K)] = \int_{[-\infty,\infty]^d} \nu(\mathbf{y}^{-1} \cdot K)\, \mathbb{P}_{\mathbf{Y}}(\mathrm{d}\mathbf{y}) . \tag{13}$$

Theorem 3. *Let $\mathbf{X} \in MRV(\alpha, a, \nu)$. Let $\mathbf{Y}$ be a random vector independent of $\mathbf{X}$. Assume that there exists a positive ε such that $0 < \mathbb{E}[|Y_i|^{\alpha+\epsilon}] < \infty$ for each i in $\{1, \dots, d\}$. Then the random vector $\mathbf{Y} \cdot \mathbf{X} \in MRV(\alpha, a, \nu_{\mathbf{Y}})$, i.e.*

$$t\mathbb{P}(\mathbf{Y} \cdot \mathbf{X} \in a(t)\cdot) \xrightarrow{v} \nu_{\mathbf{Y}} \ ,$$

as t tends to infinity.

The hypothesis of independence between $\mathbf{X}$ and $\mathbf{Y}$ might be restrictive. In the sequel, we generalize Theorem 3 on this point. The rest of the section is divided according to the type of dependence considered between $\mathbf{X}$ and $\mathbf{Y}$. We investigate three situations that seem meaningful in an actuarial context:

- $\mathbf{X}$ is MRV and asymptotically independent of $\mathbf{Y}$;
- $\mathbf{X}$ has independent and identically distributed (i.i.d.) regularly varying components and $\mathbf{Y}$ is predictable with respect to $\mathbf{X}$;
- $\mathbf{X}, \mathbf{Y}$ are jointly MRV and asymptotically dependent.

In each of these situations, an extension of Breiman's Theorem is stated.

3.1 Case of asymptotic independence

The first generalization of Theorem 3 is done under the condition that $\mathbf{X}$ and $\mathbf{Y}$ are asymptotically independent in the following sense. We assume that

$$t\mathbb{P}\left(\left(\frac{\mathbf{X}}{a(t)}, \mathbf{Y}\right) \in \cdot\right) \xrightarrow{v} (\nu \times L)(\cdot) \tag{14}$$

on the Borel sets of $\mathbb{E}_d \times [-\infty, \infty]^d$ where ν is a Radon measure on $\mathbb{E}_d$ not concentrated at infinity and L is a probability measure on $[-\infty, \infty]^d$. Note that the roles of $\mathbf{X}$ and $\mathbf{Y}$ are not symmetric in this definition since it specifies their independence when $\mathbf{X}$ is large only. Condition (14) implies in particular that $\mathbf{X} \in \mathrm{MRV}(\alpha, a, \nu)$ for some positive α. This case obviously contains the case of stochastic independence between $\mathbf{X}$ and $\mathbf{Y}$, L being then the distribution of $\mathbf{Y}$.

Following Maulik et al. [2002], we make the following asymptotic negligibility assumption. Assume that for some $\delta > 0$ and any i in $\{1, \dots, d\}$

$$\lim_{\varepsilon\to 0} \limsup_{t\to\infty} t\mathbb{E}\left[\left(\frac{|\mathbf{X}||Y_i|}{a(t)}\right)^{\delta} \mathbb{1}_{|\mathbf{X}|/a(t) \le \varepsilon}\right] = 0 \ , \tag{15}$$

and also that

$$\int_{[-\infty,\infty]^d} \|\mathbf{y}\|^{\alpha} L(\mathrm{d}\mathbf{y}) < \infty \ . \tag{16}$$

Let ν_L be the measure defined on $\mathbb{E}_d$ by

$$\nu_L(A) = (\nu \times L)\left(\{(\mathbf{x}, \mathbf{y}), \mathbf{x} \in \mathbf{y}^{-1} \cdot A\}\right) = \int_{[-\infty,\infty]^d} \nu(\mathbf{y}^{-1} \cdot A) L(\mathrm{d}\mathbf{y}) \ .$$

Note that if $\mathbf{Y}^*$ is a random vector with distribution L, then $\nu_L = \nu_{\mathbf{Y}^*}$.

Theorem 4. *Assume that Assumptions (14), (15) and (16) hold. Then the random vector* $\mathbf{Y} \cdot \mathbf{X} \in MRV(\alpha, a, \nu_L)$*, i.e.*

$$t\mathbb{P}(\mathbf{Y} \cdot \mathbf{X} \in a(t)\cdot) \xrightarrow{v} \nu_L \ ,$$

as t tends to infinity.

The proofs of Theorem 4 and subsequent results are postponed until Section 6. We establish in the proof of Theorem 4 a more general result on a random vector $\mathbf{MX}$, where $\mathbf{M}$ is a random matrix of size $q \times d$.

3.2 Case of predictable weights

The following generalization uses the predictable framework introduced by Hult and Samorodnitsky [2008]. In this context, we assume that the components of the random vector $\mathbf{X}$ are independent (non necessarily identically distributed) and regularly varying at infinity. Then the vector $\mathbf{X}$ is multivariate regularly varying. More precisely, the sequence $t\mathbb{P}(\mathbf{X}/a(t) \in \cdot)$ converges vaguely to a measure ν, which is concentrated on the axes

$$\nu = \sum_{j=1}^{d} \nu_j \ ,$$

where $\nu_j(A) = \nu(A \cap \delta_j)$ and $\delta_j = \{z \in \mathbb{R}^d \mid z_j \neq 0, z_i = 0, i \neq j\}$ is the j-th punctured coordinate axis.

The predictable framework of Hult and Samorodnitsky [2008] consists in assuming that there exists a filtration with respect to which the X_j's are measurable and the Y_j's are predictable. This implies in particular that for each j, X_j and Y_j are independent.

This framework is of interest in time series. An example is the EGARCH process of Nelson [1991]. This process, say $\{\zeta_j\}$, can be expressed as

$$\zeta_j = X_j Y_j \ , \quad Y_j = \exp\left\{\sum_{i=1}^{\infty} c_i \eta_{j-i}\right\} \ ,$$

and the relevant filtration is $\mathcal{F}_j = \sigma(X_i, \eta_i, i \leq j-1)$, $\{X_j\}$ and $\{\eta_j\}$ are two i.i.d. sequences, not necessarily independent of each other, $\mathbb{E}[\eta_j] = 0$, $\mathrm{Var}(\eta_j) = 1$ and $\sum_{i=1}^{\infty} c_i^2 < \infty$. The process $\{Y_j\}$ is called the volatility. The extremal properties of this process have been studied by Davis and Mikosch [2001] in the case of independence between the processes $\{X_j\}$ and $\{Y_j\}$.

Theorem 5. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors of $\mathbb{R}^d$. Assume that the components of $\mathbf{X}$ are independent (non necessarily identically distributed) and $\mathbf{X}$ is $MRV(\alpha, a, \nu)$. Suppose that there exists a filtration $\{\mathcal{F}_j\}$ such that X_j is $\mathcal{F}_{j+1}$-measurable and Y_j is $\mathcal{F}_j$-measurable. Assume moreover that there exists $\varepsilon > 0$ such that*

$$\mathbb{E}[|Y_j|^{\alpha+\varepsilon}] < \infty \quad \forall j = 1, \ldots, d \ . \tag{17}$$

Then the random vector $\mathbf{Y} \cdot \mathbf{X}$ *is multivariate regularly varying and*

$$t\mathbb{P}(\mathbf{Y} \cdot \mathbf{X} \in a(t)\cdot) \xrightarrow{v} \sum_{j=1}^{d} \mathbb{E}[Y_j^{\alpha}]\nu_j \ , \tag{18}$$

as t *tends to infinity.*

Remark 5. The assumptions of the previous theorem are weaker than those of Hult and Samorodnitsky [2008]. Their mean condition (3.1) and their assumptions (3.7)–(3.9) here reduce to (17) thanks to the finite time horizon d.

3.3 Case of joint multivariate regular variation

When neither independence nor asymptotic independence is a relevant assumption, one might be interested in extensions of Breiman's Theorem under asymptotic dependence. Specifically, we assume that $\mathbf{X}$ and $\mathbf{Y}$ are jointly multivariate regularly varying, i.e. there exist a and b such that

$$t\mathbb{P}\left(\left(\frac{X_i}{a(t)}, \frac{Y_i}{b(t)}\right)_{i=1,\dots,d} \in \cdot\right) \xrightarrow{v} \nu_{\mathbf{X},\mathbf{Y}} \ , \tag{19}$$

as t tends to infinity, for a non-null Radon measure $\nu_{\mathbf{X},\mathbf{Y}}$ on $\mathbb{E}_{2d}$. We assume that the measure $\nu_{\mathbf{X},\mathbf{Y}} \circ \Pi^{-1}$ is not identically zero. It means that there exists one index i such that the pair (X_i, Y_i) is asymptotically dependent, i.e.

$$\exists\ i \in \{1, \dots, d\} : \quad \nu^{(i)}_{\mathbf{X},\mathbf{Y}}((0,\infty]^2) > 0 \ , \tag{20}$$

where $\nu^{(i)}_{\mathbf{X},\mathbf{Y}}$ is defined as the restriction of ν to the i-th coordinates in $\mathbf{x}$ and $\mathbf{y}$. For instance when $i = 1$, $\nu^{(1)}(A) = \nu_{2d}(A \times [-\infty, \infty]^{2d-2})$ for any $A \in \mathcal{B}_2$. The normalizing functions a and b are regularly varying with indices respectively denoted by $1/\alpha$ and $1/\beta$ for some positive real numbers α and β.

Define the map $\Pi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ by $\Pi(\mathbf{x}, \mathbf{y}) = \mathbf{y} \cdot \mathbf{x}$.

Theorem 6. *Assume that Assumptions (19) and (20) hold. Then the random vector* $\mathbf{Y} \cdot \mathbf{X} \in$ *MRV*$(\alpha\beta/(\alpha+\beta), ab, \nu_{\mathbf{X},\mathbf{Y}} \circ \Pi^{-1})$*, i.e.*

$$t\mathbb{P}(\mathbf{Y} \cdot \mathbf{X} \in a(t)b(t)\cdot) \xrightarrow{v} \nu_{\mathbf{X},\mathbf{Y}} \circ \Pi^{-1} \ ,$$

as t *tends to infinity.*

In the particular case $Y_1 = \dots = Y_d$, this result can be found in Resnick [2007, Proposition 7.6].

Remark 6. Traditional models include an assumption of regular variation on the $\exp(Y_i)$'s instead of the Y_i's; for instance when the Y_i's are log-normal. As a consequence, assuming that the discount factors have heavy tails might be considered as a strong hypothesis in a usual context. However, this model yields to quite light tails for large values of β and allows dependence between the $\mathbf{X}$ and the $\mathbf{Y}$ components.

4 Application to risk measures

The link between a Breiman's type result and asymptotics for the ruin probability is highlighted in the first part of this section. The equivalents that we obtain depend on the ratio $Q(\mathbf{Y}\cdot\mathbf{X})$ which is not always explicitly known. Since numerical approximations can be done, we show some patterns in the second part of this section in the case of the asymmetric logistic measure. Finally, in the third part of this section, we derive the consequences of our results on another risk measure: the Value-at-Risk.

4.1 Ruin probability

Recall that the ruin probability associated to the model (1) is defined by

$$\psi(x,d) = \mathbb{P}\left(\max_{1\le k\le d}\sum_{i=1}^{k} X_i Y_i > x\right)$$

and that for any random vector $\mathbf{Z}$ we have defined $Q(\mathbf{Z})$ by

$$Q(\mathbf{Z}) = \lim_{x\to\infty}\frac{\mathbb{P}\left(\sum_{i=1}^{d} Z_i > x\right)}{\sum_{i=1}^{d}\mathbb{P}(Z_i > x)}$$

when the limit exists.

The next result establishes that the first equivalence of (4) holds if the random vector $(X_1Y_1,\ldots,X_dY_d)$ is MRV and if all the random variables X_i have lower tails lighter than their upper tails.

Theorem 7. *Under the assumptions of any one of Theorems 3, 4, 5 or 6, if the components of* $\mathbf{Y}$ *are nonnegative and (9) holds, then one has, when* x *tends to infinity,*

$$\psi(x,d) \sim \mathbb{P}\left(\sum_{i=1}^{d} X_i Y_i > x\right) . \tag{21}$$

The second equivalence of (4) is specific to the asymptotic independence context considered by Zhang et al. [2009] and does not hold in general. However, as soon as a Breiman's type result is valid, one can derive other asymptotics for (21) depending explicitly on the margins of $\mathbf{X}$ and on $Q(\mathbf{Y}\cdot\mathbf{X})$. We make these relations explicit in the following corollaries, which proofs are deferred to Section 6. Let us introduce the two sets $T = \{\mathbf{z}\in\bar{\mathbb{R}}^d \mid \sum_{i=1}^{d} z_i > 1\}$ and $T_i = \{\mathbf{z}\in\bar{\mathbb{R}}^d \mid z_i > 1\}$.

Corollary 8. *Under the assumptions of Theorem 3, if the components of* $\mathbf{Y}$ *are nonnegative and (9) holds, then when* x *tends to infinity,*

$$\psi(x,d) \sim Q(\mathbf{Y}\cdot\mathbf{X})\sum_{i=1}^{d}\mathbb{E}[Y_i^{\alpha}]\,\mathbb{P}(X_i > x) , \tag{22}$$

where

$$Q(\mathbf{Y}\cdot\mathbf{X}) = \frac{\nu_Y(T)}{\sum_{i=1}^d \nu_Y(T_i)} = \frac{\mathbb{E}[\nu(\mathbf{Y}^{-1}\cdot T)]}{\sum_{i=1}^d \mathbb{E}[Y_i^\alpha]\nu(T_i)} \ . \tag{23}$$

Note that for any deterministic vector $\mathbf{Y}$, the numerator $\nu_Y(T)$ of the middle term in (23) corresponds to the extreme risk index of the portfolio denoted by $\gamma_{\mathbf{Y}}$. See Mainik and Rüschendorf [2009] and references therein for complements on this risk measure.

Corollary 9. *Under the assumptions of Theorem 4, if the components of* $\mathbf{Y}$ *are nonnegative and (9) holds, then when* x *tends to infinity,*

$$\psi(x,d) \sim Q(\mathbf{Y}\cdot\mathbf{X}) \sum_{i=1}^d \mathbb{P}(X_i > x) \int_0^\infty y_i^\alpha \mathrm{d}L(\mathbf{y}) \ , \tag{24}$$

where

$$Q(\mathbf{Y}\cdot\mathbf{X}) = \frac{\nu_L(T)}{\sum_{i=1}^d \nu_L(T_i)} = \frac{\int \nu(\mathbf{y}^{-1}\cdot T)\,\mathrm{d}L(\mathbf{y})}{\sum_{i=1}^d \nu(T_i)\int y_i^\alpha\,\mathrm{d}L(\mathbf{y})} \ .$$

Remark 7. Corollary 8 recovers the results (for finite time horizon) of Zhang et al. [2009]. Indeed, under Corollary 8, as well as under Corollary 9, it can easily be proven that the components of $\mathbf{Y}\cdot\mathbf{X}$ are asymptotically independent as soon as those of $\mathbf{X}$ are asymptotically independent, so that $Q(\mathbf{Y}\cdot\mathbf{X}) = 1$. This comes from linearity properties combined with the fact that for any vector y with positive components, any $i \in \{1,\dots,d\}$ and any subset A of $\mathbb{E}_d$ one has

$$(y^{-1}\cdot A)\cap\mathbb{R}e_i = y^{-1}\cdot(A\cap\mathbb{R}e_i) \ .$$

Corollary 10. *Under the assumptions of Theorem 5, if the components of* $\mathbf{Y}$ *are nonnegative and (9) holds, then when* x *tends to infinity,*

$$\psi(x,d) \sim \sum_{i=1}^d \mathbb{E}[Y_i^\alpha]\,\mathbb{P}(X_i > x) \ . \tag{25}$$

In the latter corollary, the independence of the claim amounts X_i yields $Q(\mathbf{Y}\cdot\mathbf{X}) = \mathbf{1}$, so that actuarial and financial informations are separated in (25).

Corollary 11. *Under the assumptions of Theorem 6, one has, when* x *tends to infinity,*

$$\psi(x,d) \sim Q(\mathbf{Y}\cdot\mathbf{X}) \sum_{i=1}^d \mathbb{P}(X_iY_i > x) \ , \tag{26}$$

where

$$Q(\mathbf{Y}\cdot\mathbf{X}) = \frac{\nu_{X,Y}\circ\Pi^{-1}(T)}{\sum_{i=1}^d \nu_{X,Y}\circ\Pi^{-1}(T_i)}.$$

Let us make some comments on the constant $Q(\mathbf{Y}\cdot\mathbf{X})$. We know from Lemma 2 that under the assumptions of any of the previous corollaries, one has

$$d^{(1-\delta)_+} \leq Q(\mathbf{Y}\cdot\mathbf{X}) \leq d^{(\delta-1)_+}$$

with $\delta=\alpha$ under Corollary 8 and 9 and $\delta=\alpha\beta/(\alpha+\beta)$ under Corollary 11. Consequently, when $\delta\in(0,1]$ the control is accurate. If $\delta>1$, these bounds fail to be sharp as the dimension d grows up. However, it is possible to derive bounds depending on moments of $\mathbf{Y}$. Let $Y_{(1)}$ and $Y_{(d)}$ be the smaller and greater components of $\mathbf{Y}$. For instance, Corollary 8 yields

$$\frac{\sum_{i=1}^d \nu(T_i)}{\frac{1}{\mathbb{E}[Y_{(1)}^\alpha]}\sum_{i=1}^d \mathbb{E}[Y_i^\alpha]\nu(T_i)} Q(\mathbf{X}) \leq Q(\mathbf{Y}\cdot\mathbf{X}) \leq \frac{\sum_{i=1}^d \nu(T_i)}{\frac{1}{\mathbb{E}[Y_{(d)}^\alpha]}\sum_{i=1}^d \mathbb{E}[Y_i^\alpha]\nu(T_i)} Q(\mathbf{X})\,.$$

These bounds are particularly interesting in the fact that they allow to separate the financial information contained in $\mathbf{Y}$ and the actuarial part given by $\mathbf{X}$ and ν. The previous upper and lower bounds can also be relaxed and lead to

$$\frac{\mathbb{E}[Y_{(1)}^\alpha]}{\max_{i=1,\dots,d}\mathbb{E}[Y_i^\alpha]} Q(\mathbf{X}) \leq Q(\mathbf{Y}\cdot\mathbf{X}) \leq \frac{\mathbb{E}[Y_{(d)}^\alpha]}{\min_{i=1,\dots,d}\mathbb{E}[Y_i^\alpha]} Q(\mathbf{X})\,.$$

Analogous bounds may be obtained under the assumptions of Corollary 9:

$$\frac{\int_{[0,\infty]^d} y_{(1)}^\alpha \,\mathrm{d}L(\mathbf{y})}{\max_{i=1,\dots,d}\int_{[0,\infty]^d} y_i^\alpha \,\mathrm{d}L(\mathbf{y})} Q(\mathbf{X}) \leq Q(\mathbf{Y}\cdot\mathbf{X}) \leq \frac{\int_{[0,\infty]^d} y_{(d)}^\alpha \,\mathrm{d}L(\mathbf{y})}{\min_{i=1,\dots,d}\int_{[0,\infty]^d} y_i^\alpha \,\mathrm{d}L(\mathbf{y})} Q(\mathbf{X})\,.$$

When the limit measure ν of $\mathbf{X}$ is given, numerical bounds for $Q(\mathbf{Y}\cdot\mathbf{X})$ can thus be obtained from the preceding inequalities. This relies on the computation of the value of $Q(\mathbf{X})$, that is illustrated in the following section for the bivariate logistic dependence case.

4.2 Illustrative features of $Q(\mathbf{X})$

In this part, we illustrate the behaviour of the term $Q(\mathbf{X})$ in the bivariate asymmetric logistic case. For other examples, refer e.g. to Alink et al. [2004], Barbe et al. [2006], Kortschak and Albrecher [2009], Embrechts et al. [2009b,c]. Recall that when $d=2$, $Q(\mathbf{X})$ may be written in terms of the limit measure ν as follows:

$$Q(\mathbf{X}) = \frac{\nu(T)}{\nu(T_1)+\nu(T_2)}\,,$$

where $T=\{\mathbf{z}\in\bar{\mathbb{R}}^2 \mid z_1+z_2>1\}$ and $T_i=\{\mathbf{z}\in\bar{\mathbb{R}}^2 \mid z_i>1\}$.

We consider a random vector $\mathbf{X}=(X_1,X_2)$ with the following bivariate extreme value distribution:
- both margins are identically distributed from the Fréchet$(\mu=0,\sigma=1,\xi=\alpha)$ distribution, which means that there exists some positive α such that for any positive x one has $F_{X_i}(x)=\mathbb{P}(X_i\leq x)=\exp\left(-x^{-\alpha}\right)$;
- the dependence between the margins of $\mathbf{X}$ is characterized by a function ℓ satisfying the

conditions of a stable tail dependence function (see e.g. L1, L2 and L3 conditions of [Beirlant et al., 2004, page 257]), so that the distribution function of $\mathbf{X}$ can be written as

$$F_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}(\mathbf{X} \leq \mathbf{x}) = \exp\left(-\ell\left(x_1^{-\alpha}, x_2^{-\alpha}\right)\right).$$

We fix the dependence tail function to be the asymmetric logistic defined by

$$\ell_{\psi_1,\psi_2,r}(x_1, x_2) = (1-\psi_1)x_1 + (1-\psi_2)x_2 + \left\{(\psi_1 x_1)^{1/r} + (\psi_2 x_2)^{1/r}\right\}^r ,$$

for any $0 < r \leq 1$ and $0 \leq \psi_1, \psi_2 \leq 1$. This parametric model has been widely used, and is e.g. presented in Section 9.2.2 of Beirlant et al. [2004].

We compute $Q(\mathbf{X})$ by standard upper and lower Riemann approximations. In dimension two, this procedure is very accurate: lower and upper curves are so closed that they are indistinguishable, see Figures 1 and 2.

In the special case where $\psi_1 = \psi_2 = 1$, the distribution of $\mathbf{X}$ is the bivariate symmetric logistic model with Fréchet margins. The strength of dependence between the components of $\mathbf{X}$ is a decreasing function of r. In particular, the independence (resp. the total positive dependence) corresponds to $r = 1$ (resp. $r \to 0$). The values of $Q(\mathbf{X})$ for this model are given on Figure 1 as functions of r and α.

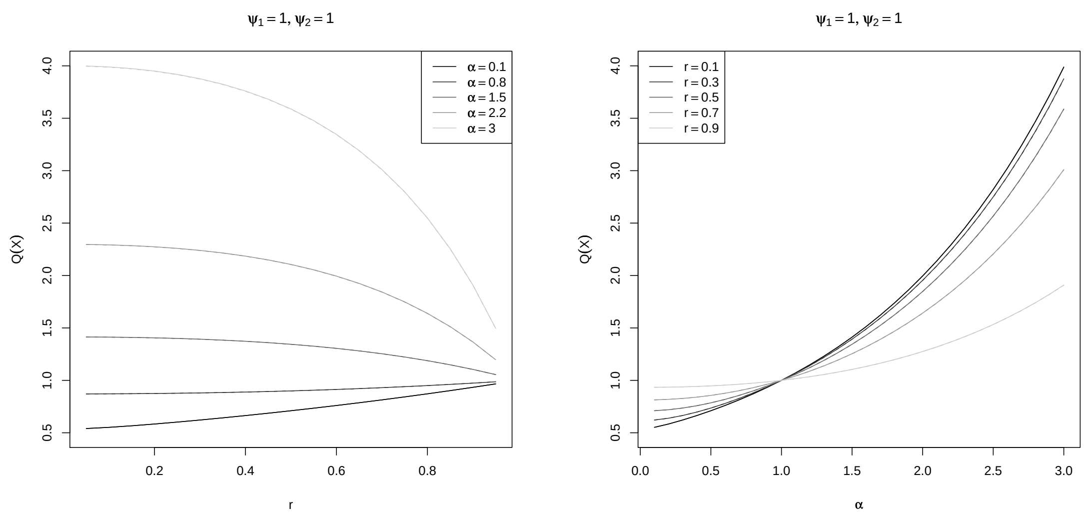

Figure 1: Values of $Q(\mathbf{X})$ as a function of r or α when $\mathbf{X}$ has the symmetric logistic dependence structure $\ell_{1,1,r}$ and Fréchet(α) margins. Left plot gives $Q(\mathbf{X})$ as a function of the dependence parameter $r \in (0,1)$, for values of α among $\{0.1, 0.8, 1.5, 2.2, 3\}$, and right plot gives $Q(\mathbf{X})$ as a function of $\alpha \in (0.1, 3)$ for values of r among $\{0.1, 0.3, 0.5, 0.7, 0.9\}$.

Several aspects of these plots were already observed in the literature for other models: $Q(\mathbf{X})$ tends to 1 as r tends to 1; $Q(\mathbf{X})$ tends to $2^{\alpha-1}$ as r tends to 0; For $\alpha < 1$ (resp. $\alpha > 1$), $Q(\mathbf{X})$ is strictly increasing (resp. decreasing) in r; For all r, $Q(\mathbf{X})$ is strictly increasing in α.

More generally, the patterns of $Q(\mathbf{X})$ when ψ_1 and ψ_2 are in $(0,1)$ are presented in Figure2. For specific values of α, ψ_1 and r, we plot $Q(\mathbf{X})$ as a function of the parameter ψ_2.

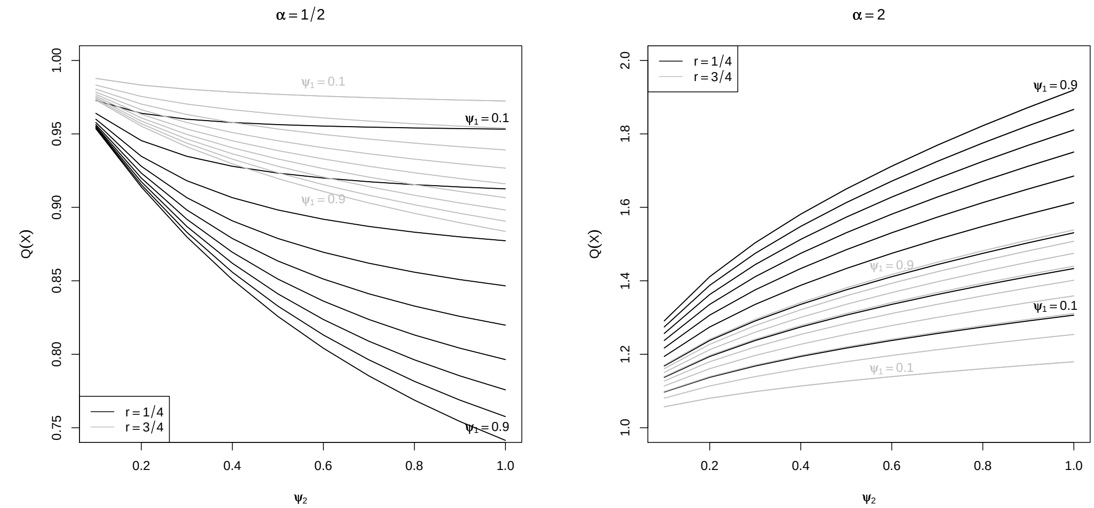

Figure 2: Values of $Q(\mathbf{X})$ as a function of ψ_2 when $\mathbf{X}$ has the asymmetric dependence structure $\ell_{\psi_1,\psi_2,r}$ and Fréchet(α) margins. Left plot (resp. right plot) gives the values of $Q(\mathbf{X})$ for $\alpha = 1/2$ (resp. $\alpha = 2$) as a function of ψ_2 for different choices of ψ_1, from top to bottom (resp. from bottom to top) $\psi_1 = 0.1, \ldots, 0.9$, and the values $r = 1/4, 3/4$.

Again we observe several elements on these plots: $Q(\mathbf{X})$ is strictly decreasing (resp. increasing) in ψ_1 or ψ_2 when $\alpha < 1$ (resp. $\alpha > 1$); For any values of ψ_1 and ψ_2, $Q(\mathbf{X})$ is strictly increasing (resp. decreasing) in r when $\alpha < 1$ (resp. $\alpha > 1$).

The computation of numerical approximations of $Q(\mathbf{X})$ can also be performed for values of d larger than two. Note however that this procedure becomes quickly time consuming when d is large.

4.3 Value-at-Risk

Another classical risk measure is the Value-at-Risk, defined as the following: Given $p \in (0,1)$ and a random variable Z, the Value-at-Risk of Z at level p, denoted by $\mathrm{VaR}_p(Z)$, is the $(1-p)$-th quantile of Z, i.e.:

$$\mathrm{VaR}_p(Z) = \inf\{z \mid \mathbb{P}(Z > z) \leq p\} .$$

An important feature of risk measures is the property of *subadditivity* (resp. *superadditivity*), which can be written for the Value-at-Risk in terms of the ratio

$$R_{\mathbf{Z}}^{(p)} = \frac{\mathrm{VaR}_p\left(\sum_{i=1}^d Z_i\right)}{\sum_{i=1}^d \mathrm{VaR}_p(Z_i)} , \tag{27}$$

where $\mathbf{Z}$ denotes the vector $(Z_1, \ldots, Z_d)^T$. We say that *subadditivity* (resp. *superadditivity*) holds for $\mathrm{VaR}_p(Z)$ if $R_{\mathbf{Z}}^{(p)} \leq 1$ (resp. > 1). Analogously, *asymptotic subadditivity* (resp. *asymptotic superadditivity*) holds for $\mathrm{VaR}_p(Z)$ if:

$$\lim_{p \to 0} R_{\mathbf{Z}}^{(p)} \leq 1 \quad (\text{resp.} > 1) .$$

The following corollary summarizes the consequences of the asymptotics obtained in the previous subsection on the Value-at-Risk, and allows in particular to discuss explicit cases where asymptotic subadditivity (resp. asymptotic superadditivity) of $\mathrm{VaR}_p(\mathbf{Y}\cdot\mathbf{X})$ holds. Such kind of comments are given by Embrechts et al. [2009a] in the case of elliptical claim amounts.

Corollary 12. *Assume that the components of* $\mathbf{Y}$ *are nonnegative and (9) holds. Then, under the assumptions of Theorem 3,*

$$\lim_{p\to 0} R^{(p)}_{\mathbf{Y}\cdot\mathbf{X}} = Q(\mathbf{Y}\cdot\mathbf{X})^{1/\alpha}\frac{\left(\sum_{i=1}^d \mathbb{E}[Y_i^\alpha]\nu(T_i)\right)^{1/\alpha}}{\sum_{i=1}^d \left(\mathbb{E}[Y_i^\alpha]\nu(T_i)\right)^{1/\alpha}}\,.$$

Under the assumptions of Theorem 4,

$$\lim_{p\to 0} R^{(p)}_{\mathbf{Y}\cdot\mathbf{X}} = Q(\mathbf{Y}\cdot\mathbf{X})^{1/\alpha}\frac{\left(\sum_{i=1}^d \int_0^\infty y_i^\alpha L(\mathrm{d}\mathbf{y})\nu(T_i)\right)^{1/\alpha}}{\sum_{i=1}^d \left(\int_0^\infty y_i^\alpha L(\mathrm{d}\mathbf{y})\nu(T_i)\right)^{1/\alpha}}\,.$$

Under the assumptions of Theorem 5,

$$\lim_{p\to 0} R^{(p)}_{\mathbf{Y}\cdot\mathbf{X}} = \frac{\left(\sum_{i=1}^d \mathbb{E}[Y_i^\alpha]\nu(T_i)\right)^{1/\alpha}}{\sum_{i=1}^d \left(\mathbb{E}[Y_i^\alpha]\nu(T_i)\right)^{1/\alpha}}\,.$$

Under the assumptions of Theorem 6,

$$\lim_{p\to 0} R^{(p)}_{\mathbf{Y}\cdot\mathbf{X}} = Q(\mathbf{Y}\cdot\mathbf{X})^{1/\alpha}\frac{\left(\sum_{i=1}^d \nu_{\mathbf{X},\mathbf{Y}}\circ\Pi^{-1}(T_i)\right)^{1/\alpha}}{\sum_{i=1}^d \left(\nu_{\mathbf{X},\mathbf{Y}}\circ\Pi^{-1}(T_i)\right)^{1/\alpha}}\,.$$

The proof of Corollary 12 is postponed until Section 6.

5 Concluding comments and discussion

In this paper, discrete time risk models with finite time horizon have been considered, and asymptotics for risk measures, as ruin probability or Value-at-Risk, have been obtained under different dependence settings for the claim amounts and discount factors. The fruitful role of the multivariate regularly varying setting has been highlighted, as well as the usefulness of Breiman's type results. This allowed to generalize (for finite time horizon) Zhang et al. [2009] result outside the asymptotic independence of the claim amounts, and outside the independence of claim amounts and discount factors.
A specific parameter $Q(\mathbf{Y}\cdot\mathbf{X})$ arises from these asymptotics, for which explicit bounds have been provided in terms of the limit measure of $\mathbf{X}$ and the characteristics of the discount factors $\mathbf{Y}$. These bounds are easily numerically computable as far as the time horizon d is not too large. Dealing with very high dimensions still represents a challenging numerical problem. Ideas coming from the algorithms developed in Arbenz et al. [2009] could be promising for this task.

Another important issue is to measure the accuracy of the approximations stated in Theorem 7 and its corollaries. Such a problem requires MRV hypotheses with second order conditions, as formulated in the univariate case by Degen et al. [2010]. This will be the scope of a future work.

6 Proofs

Proof of Lemma 1. We prove the first assertion of Lemma 1 by induction. Let K be a relatively compact set of $\mathbb{E}_{d-1}$ such that $\nu(\partial K) = 0$ and let $\epsilon > 0$. By assumption, the convergence (5) implies

$$\lim_{t\to\infty} t\mathbb{P}(a(t)^{-1}\mathbf{X} \in K \times (-\infty, -\epsilon]) = \nu(K \times (-\infty, -\epsilon]) \ .$$

Besides,

$$t\mathbb{P}(a(t)^{-1}\mathbf{X} \in K \times (-\infty, -\epsilon]) \leq t\mathbb{P}(X_d < -a(t)\epsilon) \to 0 \ ,$$

therefore $\nu(K \times (-\infty, -\epsilon]) = 0$, so the support of ν is included in $\mathbb{R}^{d-1} \times [0, \infty) \setminus \{\mathbf{0}\}$. The rest of the induction argument is along the same lines. It is then obvious that the measure $H_{\|\cdot\|}$ is concentrated on $\mathbb{S}^{d-1}_{\|\cdot\|} \cap [0, \infty]^d$. Now for simplicity, we consider the following sets

$$T_{\max} = \{\mathbf{z} \in \bar{\mathbb{R}}^d \mid \max_{1\leq k\leq d} \sum_{i=1}^{k} z_i > 1\} \ ,$$

$$T = \{\mathbf{z} \in \bar{\mathbb{R}}^d \mid \sum_{i=1}^{d} z_i > 1\} \ ,$$

$$T_+ = \{\mathbf{z} \in [0, \infty]^d \mid \sum_{i=1}^{d} z_i > 1\} \ .$$

The last assertion of Lemma 1 may be then written as the following: $\dfrac{\nu(T_{\max})}{\nu(T)} = \dfrac{\nu(T^+)}{\nu(T)} = 1$, which is obvious when ν is concentrated on the nonnegative quadrant.

□

Proof of Lemma 2. Recall that H_1 denotes the spectral measure of $\mathbf{X}$ with respect to the ℓ_1-norm. This is a probability measure concentrated on $\{w \in \mathbb{S}^{d-1}_1 \mid w_i \geq 0 \, , 1 \leq i \leq d\}$. The choice of the ℓ_1-norm yields that $\lim_{t\to\infty} t\mathbb{P}(\sum_{i=1}^d X_i > a(t)) = 1$. Besides, one gets from (8), for any $\mathbf{y} \in [0, \infty]^d$, that

$$\nu\left(\left\{\mathbf{z} \in [0, \infty]^d \setminus \{\mathbf{0}\} \mid \sum_{i=1}^{d} y_i z_i > 1\right\}\right) = \int_{\mathbb{S}^{d-1}_1} \left(\sum_{i=1}^{d} y_i w_i\right)^{\alpha} H_1(\mathrm{d}\mathbf{w}) \ ,$$

which implies in particular that

$$\lim_{t\to\infty} t\mathbb{P}(X_i > a(t)) = \int_{\mathbb{S}^{d-1}_1} w_i^{\alpha} H_1(\mathrm{d}\mathbf{w}) \ .$$

As a consequence, one gets

$$\lim_{t\to\infty} t\sum_{i=1}^{d} \mathbb{P}(X_i > a(t)) = \sum_{i=1}^{d} \int_{\mathbb{S}_1^{d-1}} w_i^{\alpha} H_1(\mathrm{d}\mathbf{w}) ,$$

so that

$$Q(\mathbf{X}) = \frac{1}{\sum_{i=1}^{d} \int_{\mathbb{S}_1^{d-1}} w_i^{\alpha} H_1(\mathrm{d}\mathbf{w})} .$$

If $\alpha > 1$, then the function $x \to x^{\alpha}$ is convex. By Jensen's inequality, we have

$$\frac{1}{d}\sum_{i=1}^{d} \int_{\mathbb{S}_1^{d-1}} w_i^{\alpha} H_1(\mathrm{d}\mathbf{w}) \geq \left\{ \frac{1}{d}\sum_{i=1}^{d} \int_{\mathbb{S}_1^{d-1}} w_i H_1(\mathrm{d}\mathbf{w}) \right\}^{\alpha} = \left\{ \frac{1}{d} \int_{\mathbb{S}_1^{d-1}} \sum_{i=1}^{d} w_i H_1(\mathrm{d}\mathbf{w}) \right\}^{\alpha} = d^{-\alpha} .$$

If $\alpha < 1$, then again by Jensen's inequality, the reverse bound holds:

$$d^{-\alpha} = \left\{ \frac{1}{d}\sum_{i=1}^{d} \int_{\mathbb{S}_1^{d-1}} w_i H_1(\mathrm{d}\mathbf{w}) \right\}^{\alpha} \geq \frac{1}{d}\sum_{i=1}^{d} \int_{\mathbb{S}_1^{d-1}} w_i^{\alpha} H_1(\mathrm{d}\mathbf{w}) .$$

On the other hand, if $\alpha > 1$, then $w^{\alpha} \leq w$ for $w \in [0, 1]$ so

$$\sum_{i=1}^{d} \int_{\mathbb{S}_1^{d-1}} w_i^{\alpha} H_1(\mathrm{d}\mathbf{w}) \leq \sum_{i=1}^{d} \int_{\mathbb{S}_1^{d-1}} w_i H_1(\mathrm{d}\mathbf{w}) = 1 .$$

The reverse inequality obviously holds for $\alpha < 1$. Gathering these bounds yields (12). The equality $Q(\mathbf{X}) = 1$ when the components of $\mathbf{X}$ are asymptotically independent follows from the fact that H_1 is then concentrated on the axes. □

Proof of Theorem 4

In order to simplify notations, we denote by $\|\cdot\|$ a given norm on an Euclidean space and for any $q \times d$ matrix M, the induced matrix norm is $\|M\| = \sup_{\|\mathbf{x}\|=1} \|M\mathbf{x}\|$. For a $q \times d$ matrix M and a set $K \subset \mathbb{R}^d$, we define

$$M^{-1} \cdot K = \{\mathbf{x} \in \mathbb{R}^d \mid M\mathbf{x} \in K\} .$$

In order to prove Theorem 4, we state and prove a more general result. We use the concept of asymptotic independence introduced by Maulik et al. [2002]. Assume that $\mathbf{M}$ is a random matrix of size $q \times d$ and $\mathbf{X} \in \mathbb{R}^d$ is a random vector satisfying

$$t\mathbb{P}\left(\left(\frac{\mathbf{X}}{a(t)}, \mathbf{M} \right) \in \cdot \right) \xrightarrow{v} (\nu \times G)(\cdot) \tag{28}$$

as t tends to infinity on $\mathbb{E}_d \times [-\infty, \infty]^{qd}$, where ν is a Radon measure on $\mathbb{E}_d$ not concentrated at infinity and G is a probability measure on $[-\infty, \infty]^{qd}$. This implies that ν is homogeneous with positive index α, say, and we can still choose the normalizing function a such that

$$\lim_{t\to\infty} t\mathbb{P}(\|\mathbf{X}\| > a(t)) = 1 .$$

Assume also that there exists $\delta > 0$ such that

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty} t\mathbb{E}\left[\left(\frac{\|\mathbf{M}\|\|\mathbf{X}\|}{a(t)}\right)^{\delta}\mathbb{1}_{\{\|\mathbf{X}\|\leq\epsilon a(t)\}}\right] = 0 , \tag{29}$$

$$\int_{[-\infty,\infty]^{qd}} \|M\|^{\alpha}\, G(\mathrm{d}M) < \infty . \tag{30}$$

Let ν_G denote the measure defined on $\mathbb{E}_d$ by

$$\begin{aligned}\nu_G(K) &= \nu\otimes G\left(\{(\mathbf{x},M) \mid M\mathbf{x}\in K\}\right)\\ &= \int_{[-\infty,\infty]^{qd}} \nu(M^{-1}\cdot K)\, G(\mathrm{d}M) = \mathbb{E}[\nu(\mathbf{M}^{*-1}\cdot K)] ,\end{aligned}$$

where $\mathbf{M}^*$ is a random matrix with distribution G.

Theorem 13. *Assume that Assumptions (28), (29) and (30) hold. Then*

$$t\mathbb{P}(a(t)^{-1}\mathbf{M}\mathbf{X}\in\cdot) \xrightarrow{v} \nu_G$$

as t tends to infinity.

Proof of Theorem 13. Let K be relatively compact in $\mathbb{E}_d$ such that $\nu_G(\partial K) = 0$. Fix some real number $s > 0$, write

$$\mathbb{P}\left(\mathbf{M}\mathbf{X}\in a(t)K\right) = \mathbb{P}\left(\mathbf{M}\mathbf{X}\in a(t)K\,,\ \|\mathbf{M}\|\leq s\right) + \mathbb{P}\left(\mathbf{M}\mathbf{X}\in a(t)K\,,\ \|\mathbf{M}\| > s\right) .$$

Since K is relatively compact in $\mathbb{E}_d$, there exists $\epsilon > 0$ such that $\|\mathbf{x}\|\geq\epsilon$ for all $\mathbf{x}\in K$. Thus $\|M\|\leq s$ and $M\mathbf{x}\in K$ imply that $\|\mathbf{x}\|\geq s^{-1}\epsilon$, thus Assumption (28) implies that

$$\lim_{t\to\infty} t\mathbb{P}\left(\mathbf{M}\mathbf{X}\in a(t)K\ ,\ \|\mathbf{M}\|\leq s\right) = \mathbb{E}\left[\nu(\mathbf{M}^{*-1}\cdot K)\mathbb{1}_{\{\|\mathbf{M}\|\leq s\}}\right] .$$

By homogeneity of ν, Condition (30) and the bounded convergence theorem, it holds that

$$\lim_{s\to\infty}\mathbb{E}[\nu(\mathbf{M}^{*-1}\cdot K)\mathbb{1}_{\{\|\mathbf{M}\|\leq s\}}] = \mathbb{E}[\nu(\mathbf{M}^{*-1}\cdot K)] .$$

Next, since $\mathbf{y}\in K$ implies $\|\mathbf{y}\|\geq\epsilon$, we have, for $s\geq 1$,

$$\begin{aligned}&\mathbb{P}(\mathbf{M}\mathbf{X}\in a(t)K\ ,\ \|\mathbf{M}\| > s)\\ &\qquad\leq \mathbb{P}\left(\|\mathbf{M}\|\|\mathbf{X}\| > \epsilon a(t)\ ,\ \|\mathbf{M}\| > s\right)\\ &\qquad\leq \mathbb{P}\left(\|\mathbf{X}\| > \epsilon a(t)\ ,\ \|\mathbf{M}\| s\right) + \mathbb{P}\left(\frac{\|\mathbf{M}\|\,\|\mathbf{X}\|}{a(t)} > \epsilon\ ,\ \frac{\|\mathbf{X}\|}{a(t)}\leq\epsilon\right) .\end{aligned}$$

By Assumption (28),

$$\lim_{t\to\infty} t\mathbb{P}\left(\|\mathbf{X}\| > \epsilon a(t)\ ,\ \|\mathbf{M}\| > s\right) = \epsilon^{-\alpha}\mathbb{P}(\|\mathbf{M}^*\| > s) ,$$

which can be made arbitrarily small by choosing s large enough. By Markov inequality and Assumption (29)

$$\limsup_{t\to\infty} t\mathbb{P}\left(\frac{\|\mathbf{M}\|\,\|\mathbf{X}\|}{a(t)} > \epsilon\ ,\ \frac{\|\mathbf{X}\|}{a(t)}\leq\epsilon\right)\leq\epsilon^{-\delta}\limsup_{t\to\infty} t\mathbb{E}\left[\left(\frac{\|\mathbf{M}\|\,\|\mathbf{X}\|}{a(t)}\right)^{\delta}\mathbb{1}_{\{\|\mathbf{X}\|\leq\epsilon a(t)\}}\right] = 0 .$$

Since s can be chosen arbitrarily large, we have proven that $\lim_{t\to\infty} t\mathbb{P}(\mathbf{M}\mathbf{X}\in a(t)K) = \mathbb{E}[\nu(\mathbf{M}^{*-1}\cdot K)]$ and this concludes the proof. ☐

Proof of Theorem 5. It is a straightforward consequence of Hult and Samorodnitsky [2008, Theorem 3.1] for $p = 1$; $A_0 = 0$ and $Z_0 = 0$; $A_j = Y_j \mathbf{e}_j$ and $Z_j = X_j$ for $j = 1, \ldots, d$; $A_j = 0$ and $Z_j = 0$ for $j > d$. □

Proof of Theorem 6. Let K be a relatively compact set in $\mathbb{E}_d$ such that $\nu_{\mathbf{X},\mathbf{Y}} \circ \Pi^{-1}(\partial K) = 0$. Then $\nu_{\mathbf{X},\mathbf{Y}}(\partial \Pi^{-1} K) = 0$, since Π is continuous, which implies that $\partial \Pi^{-1}(K) \subset \Pi^{-1}(\partial K)$. Moreover, $\Pi^{-1}(K)$ is relatively compact in $\mathbb{E}_{2d}$. To see this, we can choose some arbitrary norm and prove that for some $\xi > 0$ the set $K = \{\mathbf{x} \in \mathbb{R}^d \mid \|x\| > \xi\}$ is such that $\Pi^{-1}(K)$ is bounded away from zero in $\mathbb{E}_{2d}$. Choose for instance the euclidean norm in both $\mathbb{E}_d$ and $\mathbb{E}_{2d}$. Then

$$\|\mathbf{x} \cdot \mathbf{y}\|^2 = \sum_{i=1}^d x_i^2 y_i^2 \leq \sum_{i=1}^d (x_i^2 + y_i^2) y_i^2 \leq \sum_{i=1}^d (x_i^2 + y_i^2)^2 = \|(\mathbf{x}, \mathbf{y})\|^2 .$$

Thus, if $(\mathbf{x}, \mathbf{y}) \in \Pi^{-1}(K)$, then $\|(\mathbf{x}, \mathbf{y})\| \geq \|\mathbf{x} \cdot \mathbf{y}\| > \xi$ and this proves that $\Pi^{-1}(K)$ is relatively compact in $\mathbb{E}_{2d}$. Denote $c(t) = a(t)b(t)$. Then,

$$\lim_{t\to\infty} t\mathbb{P}(\mathbf{Y} \cdot \mathbf{X}/c(t) \in K) = \lim_{t\to\infty} t\mathbb{P}\left(\left(\frac{X_i}{a(t)}, \frac{Y_i}{b(t)}\right)_{i=1,\ldots,d} \in \Pi^{-1}(K)\right) = \nu_{\mathbf{X},\mathbf{Y}} \circ \Pi^{-1}(K) .$$

□

Proof of Theorem 7. We want to state (21). Under the assumptions of any of the theorems of Section 3, $\mathbf{Y} \cdot \mathbf{X}$ is multivariate regularly varying. Let μ denote the Radon measure associated to $\mathbf{Y} \cdot \mathbf{X}$ and a the normalizing function. Then (21) is equivalent to

$$\mu(T) = \mu(T_{\max}) = \mu(T_+) .$$

This holds as soon as the support of μ is included in $[0, \infty]^d \setminus \{\mathbf{0}\}$, which we prove now. By Lemma 1, it suffices to prove that (9) holds for the vector $\mathbf{Y} \cdot \mathbf{X}$, i.e. for each $i = 1, \ldots, d$,

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_i Y_i < -x)}{\mathbb{P}(X_i Y_i > x)} = 0 . \tag{31}$$

The latter equivalence holds under the assumptions of Theorems 3, 4 and 5 since X_i and Y_i are asymptotically independent, so that

$$\mathbb{P}(X_i Y_i > x) \sim c_i \mathbb{P}(X_i > x) \text{ and } \mathbb{P}(X_i Y_i < -x) \sim c_i \mathbb{P}(X_i < -x)$$

in each of these cases, where $c_i = \mathbb{E}[Y_i^\alpha]$ under the assumptions of Theorems 3 and 5 and $c_i = \int_0^\infty y_i^\alpha L(\mathrm{d}\mathbf{y})$ under the assumptions of Theorem 4.

Under the assumptions of Theorem 6, we must prove that if X and Y are two jointly regularly random variables such that Y is non negative and X satisfies (9), then XY also satisfies (9). This clearly holds, since for any $x > 0$,

$$0 = \lim_{t\to\infty} t\mathbb{P}(X < -a(t)x) = \nu_{X,Y}((-\infty, -x) \times [0, \infty)) .$$

Thus the support of $\nu_{X,Y}$ is included in $[0, \infty]^2$. This proves (31) and concludes the proof of (21) under the assumptions of Theorem 6, and therefore also the proof of Theorem 7. □

Proof of Corollaries 8 - 9 - 10 - 11. We need to prove respectively (22) - (24) - (25) and (26). Note that (26) follows directly from the definition of $Q(\mathbf{X} \cdot \mathbf{Y})$ and is valid in all contexts. Thus, we only need to give further equivalents of $\sum_{i=1}^{d} \mathbb{P}(X_i Y_i > x)$.

- Under the assumptions of Theorems 3 and 5, X_i and Y_i are independent, and Breiman's Theorem applies: $\mathbb{P}(X_i Y_i > x) \sim \mathbb{E}[Y_i^\alpha]\, \mathbb{P}(X_i > x)$. Under the assumptions of Theorem 5, the components of $\mathbf{Y} \cdot \mathbf{X}$ are asymptotically independent as shown by (18), so that a consequence of Lemma 2 is that $Q(\mathbf{Y} \cdot \mathbf{X}) = 1$.
- Under the assumptions of Theorem 4, it follows from conditions (14), (15) and (16) that $\mathbb{P}(X_i Y_i > x) \sim \int_0^\infty y_i^\alpha \, L(\mathrm{d}\mathbf{y}) \mathbb{P}(X_i > x)$.

□

Proof of Corollary 12. The following lemma is useful to prove Corollary 12, and is a straightforward consequence of usual properties of inverses of regularly varying functions, refer e.g. to Resnick [1987, Proposition 0.8 (vi)]. We state it here (without proof) for completeness.

Lemma 14. *Let $0 < \alpha < \infty$. If X and Y are two random variables with regularly varying upper tails with index $-\alpha$, then for $0 \le a \le \infty$,*

$$\lim_{x \to \infty} \frac{\mathbb{P}(X > x)}{\mathbb{P}(Y > x)} = a \;\Leftrightarrow\; \lim_{p \to 0} \frac{\mathrm{VaR}_p(X)}{\mathrm{VaR}_p(Y)} = a^{1/\alpha} \, .$$

Under the assumptions of any of the theorems 3, 4, 5 or 6, we want to obtain the following common expression:

$$\lim_{p \to 0} \frac{\mathrm{VaR}_p \left(\sum_{i=1}^{d} X_i Y_i \right)}{\sum_{i=1}^{d} \mathrm{VaR}_p(X_i Y_i)} = \left\{ Q(\mathbf{Y} \cdot \mathbf{X}) D \right\}^{1/\alpha} \, , \tag{32}$$

where D is defined by

$$D = \frac{\sum_{i=1}^{d} \mu(T_i)}{\left\{ \sum_{i=1}^{d} \mu(T_i)^{1/\alpha} \right\}^\alpha} \, , \tag{33}$$

in terms of $T_i = \{ \mathbf{z} \in \bar{\mathbb{R}}^d \mid z_i > 1 \}$, μ denoting the limit measure of $\mathbf{Y} \cdot \mathbf{X}$. Define the random variable Z via the relationship $\mathrm{VaR}_p Z = \sum_{i=1}^{d} \mathrm{VaR}_p(X_i Y_i)$. Lemma 14 says that (32) is equivalent to

$$\lim_{x \to \infty} \frac{\mathbb{P}(\sum_{i=1}^{d} X_i Y_i > x)}{\mathbb{P}(Z > x)} = Q(\mathbf{Y} \cdot \mathbf{X}) D \, ,$$

which is also equivalent, by definition of $Q(\mathbf{Y} \cdot \mathbf{X})$, to

$$\lim_{x \to \infty} \frac{\sum_{i=1}^{d} \mathbb{P}\left(X_i Y_i > x \right)}{\mathbb{P}(Z > x)} = D \, .$$

This limit D can be explicited using the change of variable $x = \gamma(t)$, where $\gamma(t) = a(t)$ under the assumptions of Theorems 3-4-5, and $\gamma(t) = a(t)b(t)$ under the assumptions of Theorem 6. One gets then, under any of these assumptions:

$$\lim_{t\to\infty} t\mathbb{P}(X_iY_i > \gamma(t)) = \mu(T_i)\ . \tag{34}$$

This gives the numerator announced in (33). The denominator is obtained applying twice Lemma 14. Indeed, we get first from (34) that, for any $i, j \in \{1, \dots, d\}$,

$$\lim_{p\to 0} \frac{\mathrm{VaR}_p\,(X_jY_j)}{\mathrm{VaR}_p(X_iY_i)} = \left\{\frac{\mu(T_j)}{\mu(T_i)}\right\}^{1/\alpha}\ ,$$

so that for any $i \in \{1, \dots, d\}$,

$$\lim_{p\to 0} \frac{\mathrm{VaR}_p\,(Z)}{\mathrm{VaR}_p(X_iY_i)} = \lim_{p\to 0} \sum_{j=1}^{d} \frac{\mathrm{VaR}_p\,(X_jY_j)}{\mathrm{VaR}_p(X_iY_i)} = \sum_{j=1}^{d} \left\{\frac{\mu(T_j)}{\mu(T_i)}\right\}^{1/\alpha}\ .$$

This is still equivalent, thanks again to Lemma 14, to

$$\lim_{t\to\infty} \frac{\mathbb{P}\,(Z > \gamma(t))}{\mathbb{P}(X_iY_i > \gamma(t))} = \left[\sum_{j=1}^{d} \left\{\frac{\mu(T_j)}{\mu(T_i)}\right\}^{1/\alpha}\right]^{\alpha}\ .$$

Combining the last limit with (34) yields

$$\lim_{t\to\infty} t\mathbb{P}\,(Z > \gamma(t)) = \left[\sum_{j=1}^{d} \{\mu(T_j)\}^{1/\alpha}\right]^{\alpha}\ ,$$

which is the expected denominator in (33). Checking that $\mu(T_j)$ has the different expressions announced under the assumptions of Theorems 3-4-5-6 is straightforward and ends the proof of Corollary 12.

□

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